Nonlinear Fourier Analysis of Shallow Water Waves

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Grant #: N00014-98-1-0192

LONG-TERM GOALS

The long-term goal of this research is the development of new numerical tools for obtaining the *fast nonlinear Fourier transform* of shallow-water ocean surface waves. These methods are based upon the *inverse scattering transform* (IST) a modern method of mathematical physics for *integrating* particular water wave equations. The IST allows for the description of both nonlinear unidirectional and multidirectional wave trains. The present work focuses on the development of numerical tools, on the development of data analysis procedures and on the extension of IST to higher order. Clearly the future lies in the better understanding of the nonlinear dynamics of ocean surface waves and our work addresses the practical issues of making this goal a reality.

OBJECTIVES

The primary focus is on the development of new, faster and more easily facilitated computer codes for the nonlinear Fourier analysis of shallow water, ocean surface waves. We are all familiar with traditional, linear Fourier methods as the fundamental tool for the analysis of wave data. Recent results have demonstrated that *nonlinear* Fourier analysis provides an even more useful tool for understanding large waves in shallow water. To this end it now becomes important to explore ways to extend and enhance algorithms and computer codes for nonlinear Fourier methods. There are several aspects to this problem: (1) simplification and cleaning of existing computer algorithms, (2) acceleration of algorithms (essentially searches for the *fast nonlinear Fourier transform*), (3) reduction of the large quantity of computer codes to a reasonably small number and (4) building a front-end interface to the codes to allow the typical user to access the power of nonlinear Fourier techniques without necessarily becoming an expert in the requisite mathematics.

A second aspect of this work relates to the extension of nonlinear Fourier methods to higher order. One might, for example, argue that while IST is fully nonlinear, it does not describe the behavior of the highest waves, which naturally are of great concern to ocean going vessels of all types. However, we have pioneered a new approach during the past year, which allows us to apply IST to arbitrarily large shallow water waves. The largest wave was first studied by Stokes over 130 years ago; he found the wave to have a cusped form with a 110 degree angle at the peak. We have recently shown that the highest wave can be viewed as a *single nonlinear Fourier component.* The development of numerical tools for implementing this exciting discovery is one of our major objectives.

Another objective is to develop user friendly software, which will allow the nonlinear Fourier transform to become a standard tool, implementable by non-experts in inverse scattering theory, for the analysis of shallow water wave data. To this end we are developing a *front end*, using the standard computer program Matlab. The user will be able to access IST software in a windowed environment with pull down menus and full graphics support. Based upon the current situation we expect to provide user friendly software to any computer supported by Matlab. These include the PC environment in Windows 95 or 98, Windows NT and Linux; Digital Alpha machines in Linux and

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OMB No. 0704-0188

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1. REPORT DATE **1998**	2. REPORT TYPE	3. DATES COVERED **00-00-1998 to 00-00-1998**

4. TITLE AND SUBTITLE **Nonlinear Fourier Analysis of Shallow Water Waves**	5a. CONTRACT NUMBER
	5b. GRANT NUMBER
	5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)	5d. PROJECT NUMBER
	5e. TASK NUMBER
	5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **Universit?i Torino,Dipartimento di Fisica,Via Pietro Giuria 1,10125 Torino, Italy,**	8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSOR/MONITOR'S ACRONYM(S)
	11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution unlimited

13. SUPPLEMENTARY NOTES
See also ADM002252.

14. ABSTRACT

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**	**Same as Report (SAR)**	**5**	

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std Z39-18

Unix; and Hewlett-Packard, IBM and Silicon Graphs computers. Thus, virtually every existing desktop computer will be able to run user friendly inverse scattering transform software. The goal is to complete a book, a work book and a CD-ROM which will carry the user through the theory, experimental implementation, data analysis, numerical analysis and numerical application of the latest IST technology for the time series analysis of ocean wave data.

APPROACH

There exist large classes of nonlinear partial differential wave equations, which are completely solvable by the inverse scattering transform. In particular the Korteweg-deVries equation (unidirectional shallow water wave motion) and the Kodomtsev-Petviashvilli equation (two-dimensional shallow water wave motion) are being studied. The full structure of the inverse scattering transform with periodic boundary conditions is being implemented for the time series analysis of data.

WORK COMPLETED

A number of areas have seen substantial progress during the year. These include: (1) speed enhancement in the *direct* nonlinear Fourier transform, (2) progress on understanding three wave resonances in shallow water, (3) extension of IST to higher order and the application of IST procedures to extended-order wave equations and (4) work on a user friendly front end for IST has begun.

RESULTS

We first discuss the background in which progress has been made. Recall that the inverse scattering transform allows the following nonlinear Fourier decomposition of a shallow water wave train:

$$\eta(x,t) = \sum \eta_{\text{Cnoidal Waves}}(x,t) + \eta_{\text{Nonlinear Interactions}}(x,t)$$

The cnoidal waves are just ordinary Stokes waves, which are summed to infinite order; they can be sine waves, Stokes waves and solitary waves depending upon the numerical value of a nonlinear parameter in the formulation. The above equation simply states that a shallow water wave train can be represented as a linear superposition of cnoidal waves plus nonlinear interactions among the cnoidal waves. When the waves are small the interactions disappear and the cnoidal waves reduce to sine waves; in this way the nonlinear formalism also covers linear Fourier analysis. Our major new results extend applications of IST to higher order. Generally speaking we have developed an asymptotic expansion of the Euler equations using the method of Whitham and found the following wave equation (written in normalized form):

$$u_t + \varepsilon(6uu_x + u_{xxx}) + \varepsilon^2(\alpha_1 u_{5x} + \alpha_2 uu_{xxx} + \alpha_3 u_x u_{xx} + \alpha_4 u^2 u_x) = O(\varepsilon^3)$$

This is the Korteweg-deVries equation plus higher order terms in the expansion (which is here written only to second order). Can one carry out the inverse scattering transform to arbitrarily high order for wave equations of this type? The answer, surprisingly, is in the affirmative. By using an approach based upon Lie transforms (developed by Sophius Lie about 130 years ago) one is able to relate all solutions of the above equation to the solutions of the KdV equation. What does this mean in practice? It means that the single IST spectral component in the general water wave problem can be continued up to the *highest wave*. The *single nonlinear Fourier component* we call an *extended cnoidal* wave. This is because the IST spectral component for the KdV equation is a single *cnoidal wave.* Can the extended cnoidal wave be viewed as a natural generalization of the cnoidal wave? The

answer is “yes” because by changing a nonlinear parameter the wave form varies from a simple sine wave, up to a Stokes wave, a solitary wave, a wave form which is higher and narrower than a solitary wave, and finally up to the highest wave. Any particular manifestation of this wave form is a *single component* in the nonlinear Fourier formalism. In fact one can use this procedure to analytically derive Fenton’s formula for the highest wave!

We have shown to all orders that the following nonlinear Fourier decomposition holds:

$$\eta(x,t) = \sum \eta_{\text{Extended Cnoidal Waves}}(x,t) + \eta_{\text{Nonlinear Interactions}}(x,t) + \eta_{\text{Inelastic Effects}}(x,t)$$

This is evidently the general shallow-water Fourier decomposition. It differs from the low order (cnoidal wave) decomposition in two ways. First, the nonlinear Fourier spectral component becomes the extended cnoidal wave. Second, there are small inelastic effects that need to be taken into account.

Numerical results are shown in Figures 1 and 2. In Figure 1, the space-time evolution of the KdV equation (left panel) and the extended KdV equation (right panel) are shown. One can easily see the more complex dynamics in the extended equation. To visualize how the IST can be used to study these two cases, we show in Figure 2 the IST spectrum of the KdV evolution (panel (a)) and the extended KdV evolution (panel (b)). Note that the spectral components as a function of wave number *are constants of the motion for KdV*. On the other hand, the IST spectrum varies considerably as a function of time for the extended KdV equation. A detailed explanation of these dynamics is given in the papers referenced below.

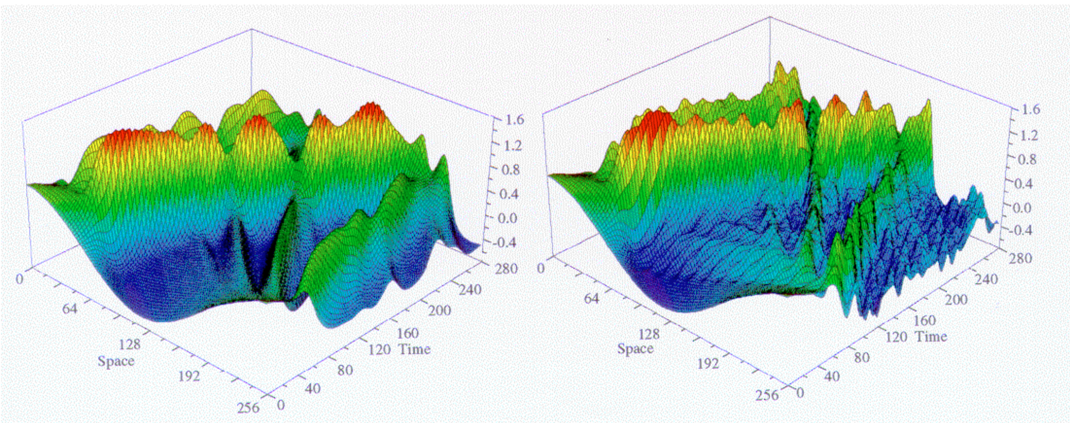

Figure 1

Finally, another interesting result, while very preliminary, is shown in Figure 3. For very large waves, near or even higher than the breaking limit, we have found the presence of *negatively displaced waves* in simulations of the extended KdV equation (see right side of Figure 3). We are unsure whether such “holes in the sea” can actually occur in the real world. In our simulations these waves are short lived, apparently subject to some instability, and they soon disappear. We show these results tentatively, with considerable trepidation, and hope to better understand wave dynamics of this kind in the near future.

IMPACT/APPLICATIONS

The impact of the present results will occur in general for the nonlinear Fourier analysis of shallow water wave trains, the analysis of internal wave trains and acoustic waves on the continental shelf, the design of floating surface and subsurface vessels, the fatigue life of tethered vehicles, etc.

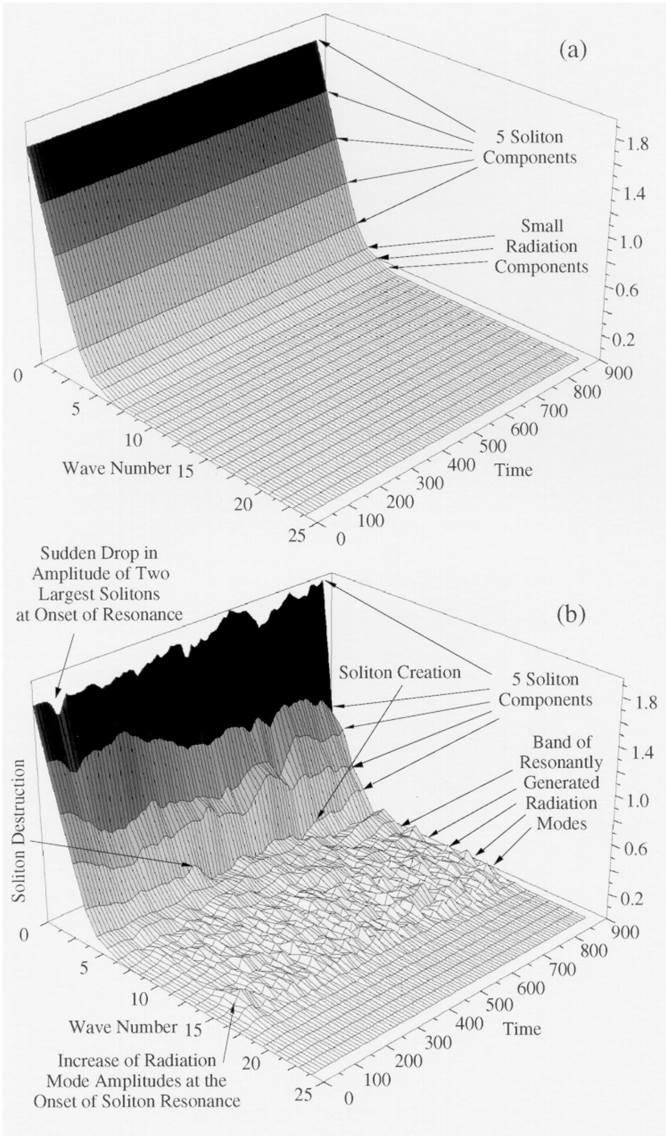

Figure 2

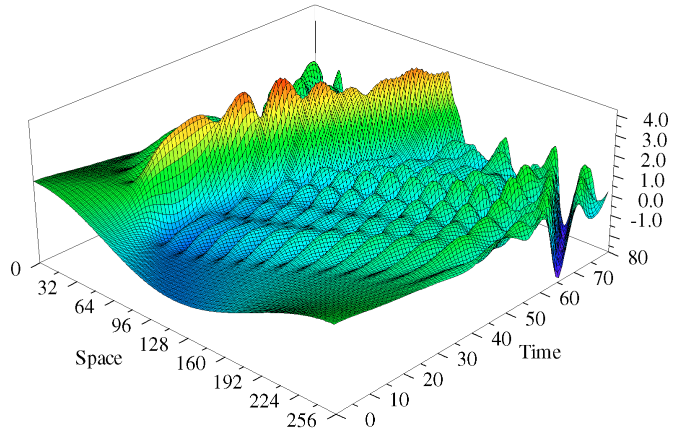

Figure 3

TRANSITIONS

Many aspects for the design of floating surface and subsurface vehicles relate to a vessel's response to the local wave forcing. In conventional analyses the *frequency response function* is computed on the basis of a sine-wave-in, sine-wave-out analysis, e.g., each sine wave in the wave field produces a sine wave response in the vessel motion. The response function is then the ratio of the amplitudes of the output to input waves at a particular frequency. In the (IST) *cnoidal wave basis* (and higher order extensions) reported on herein the spectral components are (extended) cnoidal waves and not sine waves. This idea leads to the notion of a *nonlinear response function*.

RELATED PROJECTS

An intimate relationship between my results and other projects exists because the sea surface provides a major forcing input to many kinds of offshore activities, including the dynamics of floating and drilling vessels, barges, risers and tethered vehicles. The present work leads to a nonlinear representation of the sea surface forcing and vessel response.

PUBLICATIONS

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